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Model-Free Observer Backstepping Control Design for Nonlinear Systems in Strict Feedback Form

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***Abstract*—A model-free control system design for nonlinear system in strict feedback form is presented. A type of neuro-fuzzy systems is used as system identifier. A Luenberger-type observer is designed from the identifier structure. The controller is based on backstepping and Lyapunov design schemes. Variable structure control is added to deal with uncertainties arising from estimation errors. Together, this control system is capable of controlling a system where system model is not known and only output is measurable. However, some assumptions on the actual system are required. A tracking problem example is provided together with stability proofs of the closed loop signals.**

I. INTRODUCTION

ONLY recently, researchers have begun to incorporate intelligent systems into control design. One of the earliest works appeared in [1] where multilayer neural networks were used to identify the whole plant off-line and control design was based on inverse dynamics and PD control. Reference [2] has a good collection of this control scheme and more sophisticated intelligent systems were used in [3], [4]. Reference [5] uses three-layer neural networks to identify part of the plant online with backstepping control. However part of the plant must be known. To overcome this difficulty, radial basis function networks are used to identify desired control input as in [6] with some requirements imposed on the plant.

In this paper, we present a control system as in Fig. 1. We use a type of neuro-fuzzy system called adaptive neuro-fuzzy inference system (ANFIS) as was introduced in [7] to identify the plant in real time. This intelligent system was proved to be a universal approximator in [8]. Fig. 2 depicts an ANFIS. Observer is designed from identifier structure and is the same type of observer as presented in [9]. Estimated states are fed to controller which is a composition of backstepping controller, variable structure controller and three-layer neural networks direct controller. Together, this control system is capable of controlling a system with minimum knowledge where only output is measurable.

The paper is organized as follows. Section II contains system description, general assumptions, and identifier design. Section III presents observer design. Section IV is controller design. A simulation example is given in Section V. Conclusion is in Section VI.

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II. SYSTEM DESCRIPTION AND IDENTIFIER DESIGN

Definition 1: We denote by $\|\bullet\|$ any suitable norm. When it is required to be specific we denote any p-norm by $\|\bullet\|_p$. The symbol $\|\bullet\|_F$ denotes the Frobenius norm.

Consider the system in strict feedback form:

$$\begin{aligned}\dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad 1 \le i \le m-1,\\ \dot{x}_m &= f_m(\bar{x}_m) + g_m(\bar{x}_m)u, \\ y &= x_1,\end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}, u \in \mathbb{R}, y \in \mathbb{R}, \bar{x}_i = \{x_1,\ldots,x_i\}, f_i(\bullet), g_i(\bullet)$ are unknown smooth functions. Only output y is measurable. Since $f_i(\bullet), g_i(\bullet)$ are smooth, each of them can be approximated by an ANFIS in Fig. 2 to any arbitrary accuracy. Suppose, there are *i* inputs and *j* rules. $\mu_{i,j}(x_i)$ represents a bell-shaped membership function. W_k is the multiplication of $\mu_{i,j}(x_i)$. $\bar{W}_k$ is given by

$$\bar{W}_k = W_k / \left(\sum_{l=1}^{j} W_l\right), \; k = 1,\ldots, j.$$

The output of ANFIS is given by

$$f = \sum_{k=1}^{j} \bar{W}_k \left(v_{k,i+1} + \sum_{l=1}^{i} v_{k,l} x_l\right), \tag{2}$$

where $v_k = \left[v_{k,1}, v_{k,2},\ldots, v_{k,i+1}\right] \in \mathbb{R}^{i+1}$ is a vector contain consequent parameters. Various learning schemes can be seen in [8]. In this paper, we fix parameters of the membership function and adjust only consequent parameters online using recursive least square (RLS) method.

Consider the first subsystem in (1). Suppose first that all states are measurable, we have

$$\dot{x}_1 = \hat{f}_1(x_1) + \hat{g}_1(x_1)x_2, \tag{3}$$

where $\hat{f}_1, \hat{g}_1$ are estimated functions of f_1, g_1, respectively. Suppose, for convenience, we use *J* fuzzy rules for both $\hat{f}_1, \hat{g}_1$. From (2), we have

$$\hat{f}_1 = \sum_{k=1}^{J} \bar{W}_{k,f_1}\left(v_{k,2,f_1} + v_{k,1,f_1}x_1\right),\; \hat{g}_1 = \sum_{k=1}^{J} \bar{W}_{k,g_1}\left(v_{k,2,g_1} + v_{k,1,g_1}x_1\right).$$

Equation (3) becomes

$$\dot{x}_1 = \sum_{k=1}^{J}\left(\bar{W}_{k,f_1}x_1\right)v_{k,1,f_1} + \sum_{k=1}^{J}\left(\bar{W}_{k,f_1}\right)v_{k,2,f_1} + \sum_{k=1}^{J}\left(\bar{W}_{k,g_1}x_1x_2\right)v_{k,1,g_1} + \sum_{k=1}^{J}\left(\bar{W}_{k,g_1}x_2\right)v_{k,2,g_1},$$

which can be put in the following regression form

$$Y = \Phi\Theta,$$

where $Y = [\dot{x}_1]$,

$$\Phi = \left[\bar{W}_{1,f_1}x_1, \bar{W}_{1,f_1}, \ldots, \bar{W}_{J,f_1}x_1, \bar{W}_{J,f_1}, ,\bar{W}_{1,g_1}x_1x_2, \bar{W}_{1,g_1}x_2, \ldots, \bar{W}_{J,g_1}x_1x_2, \bar{W}_{J,g_1}x_2\right],$$

$$\Theta = \left[v_{1,1,f_1}, v_{1,2,f_1}, \ldots, v_{J,1,f_1}, v_{J,2,f_1}, v_{1,1,g_1}, v_{1,2,g_1}, \ldots, v_{J,1,g_1}, v_{J,2,g_1}\right]^T.$$

Recursive least square equations are given by

$$\Theta_{i+1} = \Theta_i + P_{i+1}\Phi_{i+1}\left(Y_{i+1}^T - \Phi_{i+1}^T\Theta_i\right),$$

$$P_{i+1} = \frac{1}{\lambda}\left(P_i - \frac{P_i\Phi_{i+1}\Phi_{i+1}^T P_i}{\lambda + \Theta_{i+1}^T P_i \Theta_{i+1}}\right),$$

where $0 < \lambda \le 1$ is exponential forgetting factor. Since $\lim_{\alpha\to\infty} I/\alpha = 0$, we can select $P_0 = \alpha I$, where α is a positive large number. Θ_0 is set equal to a zero vector for convenience. Details of the RLS algorithm as well as some practical issues can be found in [10]. Identification of the remaining subsystems in (1) can be done similarly.

Since states are not measurable, all inputs and outputs to the neuro-fuzzy system are from estimated states which will be obtained from an algorithm described in Section III. We then have the estimated system as follows:

$$\begin{aligned} \dot{\xi}_i &= \hat{f}_i\left(\bar{\xi}_i\right) + \hat{g}_i\left(\bar{\xi}_i\right)\xi_{i+1}, \quad 1 \le i \le m-1, \\ \dot{\xi}_m &= \hat{f}_m\left(\bar{\xi}_m\right) + \hat{g}_m\left(\bar{\xi}_m\right)u, \\ \zeta &= \xi_1, \end{aligned} \tag{4}$$

where f_i, g_i are replaced by their corresponding estimates. ξ_i and ζ are states and output of the identified system, respectively.

III. Observer Design

From (1), we have the sequence of output derivatives:

$$\begin{aligned} y &= x_1, \\ \dot{y} &= f_1(x_1) + g_1(x_1)x_2, \\ &\vdots \\ y^{(m-1)} &= \varphi_{m-1}(x_1, x_2, \ldots, x_m). \end{aligned}$$

This can be put in the following form:

$$\begin{aligned} y_e &\triangleq \left[y_{e_1} \quad y_{e_2} \quad \ldots \quad y_{e_m}\right]^T = \left[y \quad \dot{y} \quad \ldots \quad y^{(m-1)}\right]^T, \\ &\triangleq H(\bar{x}_m) = \left[x_1 \quad \varphi_1(\bar{x}_2) \quad \ldots \quad \varphi_{m-1}(\bar{x}_m)\right]^T, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \varphi_1(\bar{x}_2) &= f_1(x_1) + g_1(x_1)(x_2), \\ &\vdots \\ \varphi_{m-1}(\bar{x}_m) &= \sum_{j=1}^{m-1}\frac{\partial\varphi_{m-2}}{\partial x_j}\left[f_j(\bar{x}_j) + g_j(\bar{x}_j)x_{j+1}\right]. \end{aligned}$$

$H(\bar{x}_m)$ is thus the mapping relating the first $m-1$ derivatives of the output y to the states of the system. Similarly we obtain (5) for the estimated system (4) as follows:

$$\begin{aligned} \zeta_e &\triangleq \left[\zeta_{e_1} \quad \zeta_{e_2} \quad \ldots \quad \zeta_{e_m}\right]^T = \left[\zeta \quad \dot{\zeta} \quad \ldots \quad \zeta^{(m-1)}\right]^T, \\ &\triangleq \hat{H}(\bar{\xi}_m) = \left[\xi_1 \quad \psi_1(\bar{\xi}_2) \quad \ldots \quad \psi_{m-1}(\bar{\xi}_m)\right]^T. \end{aligned}$$

Assumption 1: System (1) and estimated system (4) are uniformly completely observable, i.e., the mapping $H(\bar{x}_m)$ and $\hat{H}\left(\bar{\xi}_m\right)$ are invertible with respect to $\bar{x}_m$ and $\bar{\xi}_m$, and their inverses, $\bar{x}_m = H^{-1}(y_e)$ and $\bar{\xi}_m = \hat{H}^{-1}\left(\zeta_e\right)$, are smooth. Moreover, if $\left\|y_e - \zeta_e\right\| \le \varepsilon_1$ for some $\varepsilon_1 > 0$, then $\left\|H^{-1}\left(y_e\right) - \hat{H}^{-1}\left(\zeta_e\right)\right\| \le \varepsilon_2$ for some finite $\varepsilon_2 > 0$.

The *m*-order derivative of the output is as follows:

$$\begin{aligned} y^{(m)} &= \sum_{j=1}^{m-1}\frac{\partial\varphi_{m-1}}{\partial x_j}\left[f_j(\bar{x}_j) + g_j(\bar{x}_j)x_{j+1}\right] \\ &\quad + \frac{\partial\varphi_{m-1}}{\partial x_m}\left[f_m(\bar{x}_m) + g_m(\bar{x}_m)u\right] \\ &= \alpha(y_e) + \beta(y_e)u. \end{aligned} \tag{6}$$

From (5) and (6), we have

$$\dot{y}_e = Ay_e + B\left[\alpha(y_e) + \beta(y_e)u\right] \tag{7}$$

where

$$A = \begin{bmatrix} 0 & & \\ \vdots & I & \\ 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix}0 & \ldots & 0 & 1\end{bmatrix}^T.$$

Similarly, we have

$$\dot{\zeta}_e = A\zeta_e + B\left[\hat{\alpha}(\zeta_e) + \hat{\beta}(\zeta_e)u\right].$$

Replacing ξ_i with its estimated state, $\hat{x}_i$, we have

$$\begin{aligned} \hat{\zeta}_e &\triangleq \left[\hat{\zeta}_{e_1} \quad \hat{\zeta}_{e_2} \quad \ldots \quad \hat{\zeta}_{e_m}\right]^T, \\ &\triangleq \hat{H}(\bar{\hat{x}}_m) = \left[\hat{x}_1 \quad \psi_1(\bar{\hat{x}}_2) \quad \ldots \quad \psi_{m-1}(\bar{\hat{x}}_m)\right]^T. \end{aligned}$$

Assumption 2: The differences between functions f_i, g_i in (1) and their estimates $\hat{f}_i, \hat{g}_i$ in (4) are bounded.

Consider an observer for the estimated system (4) as

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \vdots \\ \dot{\hat{x}}_m \end{bmatrix} = \begin{bmatrix} \hat{f}_1\left(\hat{x}_1\right) + \hat{g}_1\left(\hat{x}_1\right)\hat{x}_2 \\ \hat{f}_2\left(\bar{\hat{x}}_2\right) + \hat{g}_2\left(\bar{\hat{x}}_2\right)\hat{x}_3 \\ \vdots \\ \hat{f}_m\left(\bar{\hat{x}}_m\right) + \hat{g}_m\left(\bar{\hat{x}}_m\right)u \end{bmatrix} + \left[\frac{\partial\hat{H}\left(\bar{\hat{x}}_m\right)}{\partial\bar{\hat{x}}_m}\right]^{-1}\varepsilon^{-1}L\left[y - \hat{\zeta}\right],$$

$$\hat{\zeta} = \hat{x}_1, \tag{8}$$

where $\varepsilon = diag\left[\eta, \eta^2, \ldots, \eta^m\right]$, $0 < \eta \le 1$ and

$\left[\partial\hat{H}(\hat{\bar{x}}_m)/\partial\hat{\bar{x}}_m\right]$ is the Jacobian of $\hat{H}$. η is a design parameter and $L=[l_1,l_2,\ldots,l_m]^T$ is such that $s^m+l_1s^{m-1}+\ldots+l_m$ is a Hurwitz polynomial.

To show stability properties of the observer estimation error dynamics we proceed as follows.

For $i=1$,

$$\dot{\hat{\zeta}}_{e_1}=\hat{\zeta}_{e_2}+I_1\left[\partial\hat{H}(\hat{\bar{x}}_m)/\partial\hat{\bar{x}}_m\right]^{-1}\varepsilon^{-1}L\left[y-\hat{\zeta}\right].$$

For $2\le i\le m-1$,

$$\dot{\hat{\zeta}}_{e_i}=\hat{\zeta}_{e_{i+1}}+\sum_{j=1}^{i}\left(\partial\psi_{i-1}/\partial\hat{x}_j\right)\left[I_j\left[\partial\hat{H}(\hat{\bar{x}}_m)/\partial\hat{\bar{x}}_m\right]^{-1}\varepsilon^{-1}L\left[y-\hat{\zeta}\right]\right].$$

For $i=m$,

$$\dot{\hat{\zeta}}_{e_m}=\hat{\alpha}(\hat{\zeta}_e)+\hat{\beta}(\hat{\zeta}_e)u+\sum_{j=1}^{m}\partial\psi_{m-1}/\partial\hat{x}_j\left[I_j\left[\partial\hat{H}(\hat{\bar{x}}_m)/\partial\hat{\bar{x}}_m\right]^{-1}\varepsilon^{-1}L\left[y-\hat{\zeta}\right]\right],$$

where $I_j\in\mathbb{R}^{1\times m}$ is a row vector whose j^{th}-element is 1 and 0 otherwise. We then can write the above in matrix form as

$$\dot{\hat{\zeta}}_e=A\hat{\zeta}_e+B\left[\hat{\alpha}(\hat{\zeta}_e)+\hat{\beta}(\hat{\zeta}_e)u\right]+\varepsilon^{-1}L\left[y-\hat{\zeta}\right]. \quad (9)$$

Define the observer error, $\tilde{\zeta}_e=\hat{\zeta}_e-y_e$.

Let $C\in\mathbb{R}^{1\times m}=[1,0,\ldots,0]$, then the observer error dynamics are given by

$$\dot{\tilde{\zeta}}_e=(A-\varepsilon^{-1}LC)\tilde{\zeta}_e+B[\hat{\alpha}(\hat{\zeta}_e)+\hat{\beta}(\hat{\zeta}_e)u-\alpha(y_e)-\beta(y_e)u].$$

Define another coordinate transformation:

$$\tilde{v}=\varepsilon'\tilde{\zeta}_e,\ \ \varepsilon'\triangleq diag\left[\frac{1}{\eta^{m-1}},\frac{1}{\eta^{m-2}},\ldots,1\right].$$

In the new coordinates, the error dynamics become

$$\dot{\tilde{v}}=\frac{1}{\eta}(A-LC)\tilde{v}+B[\hat{\alpha}(\hat{\zeta}_e)+\hat{\beta}(\hat{\zeta}_e)u-\alpha(y_e)-\beta(y_e)u].$$

By proper choice of L, $A-LC$ is Hurwitz. Let P be the solution to the Lyapunov equation

$$P(A-LC)+(A-LC)^TP=-I$$

and consider the Lyapunov candidate $V(\tilde{v})=\tilde{v}^TP\tilde{v}>0$. Its time derivative along the $\tilde{v}$ trajectories is

$$\dot{V}=-\tilde{v}^T\tilde{v}/\eta+2\tilde{v}^TPB[\hat{\alpha}(\hat{\zeta}_e)+\hat{\beta}(\hat{\zeta}_e)u-\alpha(y_e)-\beta(y_e)u].$$

Assumption 3: Control input u is bounded.

From *Assumption 2* and the fact that $\hat{f}_i,\hat{g}_i$ are continuous differentiable functions with respect to their arguments, we have

$$\left|\hat{\alpha}\left(\hat{\zeta}_e\right)-\alpha\left(\hat{\zeta}_e\right)\right|\le k_1,\quad \left|\hat{\beta}\left(\hat{\zeta}_e\right)-\beta\left(\hat{\zeta}_e\right)\right|\le k_2,$$

where k_1,k_2 are bounded constants. Since $\alpha(\bullet),\beta(\bullet)$ and their first-order derivatives are continuous and uniformly bounded on $[t_0,t]\times\mathbb{R}^m$, from Lemma 3.3 in [11], $\alpha(\bullet),\beta(\bullet)$ are globally Lipschitz. Together with *Assumption 3*, we then have

$$\left|\hat{\alpha}\left(\hat{\zeta}_e\right)+\hat{\beta}\left(\hat{\zeta}_e\right)u-\alpha\left(y_e\right)-\beta\left(y_e\right)u\right|\le k_3+k_4\left|\tilde{\zeta}_e\right|, \quad (10)$$

where k_3,k_4 are non-negative constants. From (10), we have

$$\dot{V}\left(\tilde{v}\right)\le-\left\|\tilde{v}\right\|^2/\eta+2k_3\left\|P\right\|\left\|\tilde{v}\right\|+2k_4\left\|P\right\|\left\|\tilde{v}\right\|^2.$$

We want to use part of $-\left\|\tilde{v}\right\|^2/\eta$ to dominate $2k_3\left\|P\right\|\left\|\tilde{v}\right\|$ for large $\left\|\tilde{v}\right\|$. We proceed by rewriting the foregoing inequality as

$$\begin{aligned}\dot{V}\left(\tilde{v}\right)&\le-(1-\theta)\left\|\tilde{v}\right\|^2/\eta-\theta\left\|\tilde{v}\right\|^2/\eta+2k_3\left\|P\right\|\left\|\tilde{v}\right\|+2k_4\left\|P\right\|\left\|\tilde{v}\right\|^2\\&\le-(1-\theta)\left\|\tilde{v}\right\|^2/\eta+2k_4\left\|P\right\|\left\|\tilde{v}\right\|^2,\quad\forall\left\|\tilde{v}\right\|\ge2k_3\left\|P\right\|\eta/\theta,\end{aligned}$$

where $0<\theta<1$. We can then choose $\eta<\bar{\eta}$ where $\bar{\eta}=\min\left\{(1-\theta)/2k_4\left\|P\right\|,1\right\}$ to make

$$\dot{V}\left(\tilde{v}\right)\le-W\left(\tilde{v}\right)=-\left\{(1-\theta)\left\|\tilde{v}\right\|^2/\eta-2k_4\left\|P\right\|\left\|\tilde{v}\right\|^2\right\}, \quad (11)$$

where $W\left(\tilde{v}\right)$ is positive definite function. We then can apply Theorem 4.18 in [11] to conclude that $\tilde{v}$ and hence $\tilde{\zeta}_e$ are globally uniformly ultimately bounded. The ultimate bound can be computed by using the fact that

$$\lambda_{\min}(P)\left\|\tilde{\zeta}_e\right\|^2\le V\left(\tilde{\zeta}_e\right)\le\lambda_{\max}(P)\left\|\tilde{\zeta}_e\right\|^2/(\eta^{2(m-1)}),$$

to be

$$b=\alpha_1^{-1}\left(\alpha_2\left(\mu\right)\right)=2k_3\left\|P\right\|\eta/\theta\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}.$$

Using (11), the all-time upper bound of the estimation error $\tilde{\zeta}_e$ can be found to be

$$\left\|\tilde{\zeta}_e\right\|\le1/(\eta^{(m-1)})\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}\left\|\tilde{\zeta}_e(0)\right\|\exp\left\{-\left\{(1-\theta)/\eta-2k_4\left\|P\right\|\right\}t/\left\{2\lambda_{\max}(P)\right\}\right\}.$$

IV. Controller Design

Assumption 4: The signs of all g_i functions are known. Without loss of generality, let $g_i>0,\quad\forall i=1,\ldots,m$.

Assumption 5: The functions g_i^{-1} exist and are smooth. Their time derivatives are bounded by known constant $g_{iU}>0$.

Three-layer neural network as in Fig. 3 is used to learn the desired control input of each subsystem. Variables in the network can be defined as follows:

$$\bar{Z}=\left[z_1,z_2,\ldots,z_n,1\right]^T\in\mathbb{R}^{n+1},$$

$$V=\left[v_1,v_2,\ldots,v_l\right]\in\mathbb{R}^{(n+1)\times l},$$

$$v_i=\left[v_{i1},v_{i2},\ldots,v_{i(n+1)}\right]^T\in\mathbb{R}^{n+1},i=1,2,\ldots,l,$$

$$S\left(V^T\bar{Z}\right)=\left[s\left(v_1^T\bar{Z}\right),s\left(v_2^T\bar{Z}\right),\ldots,s\left(v_l^T\bar{Z}\right),1\right]^T\in\mathbb{R}^{l+1},$$

$$W=\left[w_1,w_2,\ldots,w_l,w_{l+1}\right]^T\in\mathbb{R}^{l+1},$$

$$g(W,V,z_1,z_2,\ldots,z_n)=W^TS\left(V^T\bar{Z}\right)\in\mathbb{R}.$$

Assumption 6: Any smooth nonlinear function $h_i^*(\cdot)\in\mathbb{R}$ can be represented by a three-layer neural network with some constant ideal weights W_i^*,V_i^* as

$h_i^*(\bullet) = W_i^{*T} S_i(V_i^{*T}\bar{Z}_i) + \varepsilon_i$, where $\|\varepsilon_i\| < \varepsilon_{iU}$.

Assumption 7: On the compact set Ω_z, the ideal neural network weights W_i^*, V_i^* are constant and bounded by $\|W_i^*\| \le W_{iU}$, $\|V_i^*\|_F \le V_{iU}$, $i = 1,\ldots,m.$,

Lemma 1: Let $\hat{W}$ and $\hat{V}$ be the estimates of W^* and V^*, respectively. We have $\hat{W}^T S(\hat{V}^T\bar{Z}) - W^{*T} S(V^{*T}\bar{Z}) = \tilde{W}^T(\hat{S} - \hat{S}'\hat{V}^T\bar{Z}) + \hat{W}^T\hat{S}'\tilde{V}^T\bar{Z} + d_u$,

where

$$\hat{S} = S(\hat{V}^T\bar{Z}) \in \mathbb{R}^{l+1},$$

$$\hat{S}' = diag\left\{\hat{s}_1', \hat{s}_2', \ldots, \hat{s}_l', 0\right\} \in \mathbb{R}^{(l+1)\times(l+1)},$$

$$\hat{s}_i' = s'(\hat{v}_i^T\bar{Z}) = \left.\frac{d[s(z_a)]}{dz_a}\right|_{z_a = \hat{v}_i^T\bar{z}} \in \mathbb{R}, i = 1,2,\ldots,l,$$

$$s(z_i) = 1/(1+e^{-z_i}), \forall z_i \in \mathbb{R}.$$

The residual term d_u is bounded by

$$|d_u| \le \|V^*\|_F \left\|\bar{Z}\hat{W}^T\hat{S}'\right\|_F + \|W^*\|\left\|\hat{S}'\hat{V}^T\bar{Z}\right\| + |W^*|_1.$$

Proof: See chapter 3 of [6].

The tracking controller is designed based on backstepping scheme and proceeds according to the following steps.

Step 1: Let $z_1 = \hat{x}_1 - x_{1d} = x_1 + \tilde{x}_1 - x_{1d}$. We have $\dot{z}_1 = f_1 + g_1 x_2 + \dot{\tilde{x}}_1 - \dot{x}_{1d}$. Suppose we know f_1, g_1, the virtual control based on backstepping scheme has the form

$$x_{2d}^* = -c_1 z_1 - g_1^{-1}\left(f_1 - \dot{x}_{1d} + \dot{\tilde{x}}_1\right). \tag{12}$$

Since f_1, g_1 are unknown, the unknown part of (12), $h_1^*(\bar{Z}_1) \triangleq g_1(x_1)^{-1}\left[f_1(x_1) - \dot{x}_{1d} + \dot{\tilde{x}}_1\right]$, will be replaced by a three-layer neural network. From the observer design section, we know that $\tilde{x}_1$ is bounded at all times and we can select η to make $\tilde{x}_1 \to 0$ and hence $\dot{\tilde{x}}_1 \to 0$ arbitrarily fast. Therefore, we let inputs to the neural network be $\bar{Z}_1 = [\hat{x}_1, \dot{x}_{1d}, 1]$. We have

$$x_{2d}^* = -c_1 z_1 - W_1^{*T} S_1(V_1^{*T}\bar{Z}_1) - \varepsilon_1$$

Since W_1^*, V_1^* are unknown, let $\hat{W}_1$ be the estimate of W_1^* and $\hat{V}_1$ be the estimate of V_1^*. Introduce the error variable $z_2 = \hat{x}_2 - x_{2d}$ and choose the virtual control

$$x_{2d} = -c_1 z_1 - \hat{W}_1^T S_1(\hat{V}_1^T\bar{Z}_1) + u_{2dvsc}.$$

We add a variable structure control term u_{2dvsc} to provide robustness against estimation error $\tilde{x}_2$, approximation error ε_1 and residual term d_{u1}. From *Assumption 6* and *Lemma 1*, we have

$$|d_{u1}| + |\varepsilon_1| + |\tilde{x}_2| \le K_1^{*T}\varphi_1$$

where

$$K_1^* = \left[\|V_1^*\|_F,\ \|W_1^*\|,\ \|W_1^*\|_1 + \varepsilon_{1U} + \tilde{x}_{2U}\right]^T,$$

$$\varphi_1 = \left[\left\|\bar{Z}_1\hat{W}_1^T\hat{S}_1'\right\|_F,\ \left\|\hat{S}_1'\hat{V}_1^T\bar{Z}_1\right\|,\ 1\right]^T,$$

and $\tilde{x}_{2U}$ is the upper bound of the observer error $\tilde{x}_2 = \hat{x}_2 - x_2$. To avoid control chattering and to be able to rigorously prove the stability property of the closed-loop system without having to worry about functions with discontinuous right-hand sides, the smooth variable structure control law is used and is given by

$$u_{2dvsc} = -\hat{K}_1^T\bar{\varphi}_1,$$

$$\bar{\varphi}_1 = \begin{bmatrix} \left\|\bar{Z}_1\hat{W}_1^T\hat{S}_1'\right\|_F \dfrac{2}{\pi}\arctan\left(\dfrac{z_1}{\eta_1}\left\|\bar{Z}_1\hat{W}_1^T\hat{S}_1'\right\|_F\right) \\ \left\|\hat{S}_1'\hat{V}_1^T\bar{Z}_1\right\| \dfrac{2}{\pi}\arctan\left(\dfrac{z_1}{\eta_1}\left\|\hat{S}_1'\hat{V}_1^T\bar{Z}_1\right\|\right) \\ \dfrac{2}{\pi}\arctan\left(\dfrac{z_1}{\eta_1}\right) \end{bmatrix}, \tag{13}$$

where $\hat{K}_1$ is the estimate of K_1^*. The $\dot{z}_1$ equation becomes

$$\dot{z}_1 = g_1([z_2 - c_1 z_1 + \varepsilon_1 - \tilde{W}_1^T(\hat{S}_1 - \hat{S}_1'\hat{V}_1^T\bar{Z}_1) - \hat{W}_1^T\hat{S}_1'\tilde{V}_1^T\bar{Z}_1 - d_{u1} - \hat{K}_1^T\bar{\varphi}_1] + \tilde{x}_2).$$

Choose the Lyapunov function as

$$V_1 = \frac{1}{2}g_1^{-1}z_1^2 + \frac{1}{2}\tilde{W}_1^T\Gamma_{w1}^{-1}\tilde{W}_1 + \frac{1}{2}tr\left\{\tilde{V}_1^T\Gamma_{v1}^{-1}\tilde{V}_1\right\} + \frac{1}{2}\tilde{K}_1^T\Gamma_{k1}^{-1}\tilde{K}_1,$$

where $\Gamma_{w1}, \Gamma_{v1}, \Gamma_{k1} > 0$. Consider the following adaptation laws:

$$\begin{aligned} \dot{\hat{W}}_1 &= \dot{\tilde{W}}_1 = \Gamma_{w1}[(\hat{S}_1 - \hat{S}_1'\hat{V}_1^T\bar{Z}_1)z_1 - \sigma_{w1}\hat{W}_1], \\ \dot{\hat{V}}_1 &= \dot{\tilde{V}}_1 = \Gamma_{v1}[\bar{Z}_1\hat{W}_1^T\hat{S}_1'z_1 - \sigma_{v1}\hat{V}_1], \\ \dot{\hat{K}}_1 &= \dot{\tilde{K}}_1 = \Gamma_{k1}[\bar{\varphi}_1 z_1 - \sigma_{k1}\hat{K}_1]. \end{aligned} \tag{14}$$

The derivative $\dot{V}_1$ can be found as follows:

$$\dot{V}_1 \le z_1 z_2 - c_1 z_1^2 + K_1^{*T}\varphi_1|z_1| - z_1\hat{K}_1^T\bar{\varphi}_1 - \tilde{W}_1^T\sigma_{w1}\hat{W}_1 - tr\left\{\tilde{V}_1^T\sigma_{v1}\hat{V}_1\right\} + \tilde{K}_1^T\bar{\varphi}_1 z_1 - \tilde{K}_1^T\sigma_{k1}\hat{K}_1.$$

Using the facts that

$$0 \le |\alpha| - \alpha\frac{2}{\pi}\arctan\left(\frac{\alpha}{\eta}\right) \le 0.2785\eta, \quad \forall\alpha \in \mathbb{R},$$

$$2\tilde{W}^T\hat{W} = \|\tilde{W}\|^2 + \|\hat{W}\|^2 - \|W^*\|^2 \ge \|\tilde{W}\|^2 - \|W^*\|^2,$$

$$2tr\left\{\tilde{V}^T\hat{V}\right\} = \|\tilde{V}\|_F^2 + \|\hat{V}\|_F^2 - \|V^*\|_F^2 \ge \|\tilde{V}\|_F^2 - \|V^*\|_F^2,$$

we have

$$\dot{V}_1 \le z_1 z_2 - c_1 z_1^2 - \frac{\sigma_{w1}}{2}\|\tilde{W}_1\|^2 - \frac{\sigma_{v1}}{2}\|\tilde{V}_1\|_F^2 - \frac{\sigma_{k1}}{2}\|\tilde{K}_1\|^2 + \xi_1,$$

where

$$\xi_1 = 0.2785\eta_1(\|V_1^*\|_F + \|W_1^*\| + \|W_1^*\|_1 + \varepsilon_{1U} + \tilde{x}_{2U}) + \frac{\sigma_{w1}}{2}\|W_1^*\|^2 + \frac{\sigma_{v1}}{2}\|V_1^*\|_F^2 + \frac{\sigma_{k1}}{2}\|K_1^*\|^2. \tag{15}$$

Step i $(2 \le i \le m-1)$: Let $z_i = \hat{x}_i - x_{id}$, we obtain

$$\dot{z}_i = g_i([z_{i+1} - z_{i-1} - c_i z_i + \varepsilon_i - \tilde{W}_i^T(\hat{S}_i - \hat{S}_i'\hat{V}_i^T\bar{Z}_i) - \hat{W}_i^T\hat{S}_i'\tilde{V}_i^T\bar{Z}_i - d_{ui} - \hat{K}_i^T\bar{\varphi}_i] + \tilde{x}_{i+1}).$$

Choose the Lyapunov function as

$$V_i = V_{i-1} + \frac{1}{2} g_i^{-1} z_i^2 + \frac{1}{2}\tilde{W}_i^T \Gamma_{wi}^{-1} \tilde{W}_i + \frac{1}{2} tr\left\{\tilde{V}_i^T \Gamma_{vi}^{-1} \tilde{V}_i\right\} + \frac{1}{2}\tilde{K}_i^T \Gamma_{ki}^{-1}\tilde{K}_i,$$

where $\Gamma_{wi}, \Gamma_{vi}, \Gamma_{ki} > 0$. The weight adaptation laws have the same format as (14), we have

$$\dot{V}_i \le z_i z_{(i+1)} + \sum_{k=1}^{i}\left\{-c_k z_k^2 - \frac{\sigma_{wk}}{2}\left\|\tilde{W}_k\right\|^2 - \frac{\sigma_{vk}}{2}\left\|\tilde{V}_k\right\|_F^2 - \frac{\sigma_{kk}}{2}\left\|\tilde{K}_k\right\|^2 + \xi_k\right\},$$

where ξ_k is in the same format as (15). The term $z_i z_{(i+1)}$ will be cancelled in the next step.

Step m: This is the last step. Let $z_m = \hat{x}_m - x_{md}$. The actual control is

$$u = -z_{(m-1)} - c_m z_m - \hat{W}_m^T S_m(\hat{V}_m^T \bar{Z}_m) + u_{vsc}.$$

The variable structure control term has the same format as (13). We then have

$$\dot{z}_m = g_m([-z_{m-1} - c_m z_m + \varepsilon_m - \tilde{W}_m^T(\hat{S}_m - \hat{S}_m' \hat{V}_m^T \bar{Z}_m) - \hat{W}_m^T \hat{S}_m' \tilde{V}_m^T \bar{Z}_m - d_{um} - \hat{K}_m^T \bar{\varphi}_m]).$$

The Lyapunov function is as in the previous step, with $i = m.$ Using similar weight adaptation laws as (14), we can find the derivative $\dot{V}_m$ to be

$$\dot{V}_m \le \sum_{k=1}^{m}\left\{-c_k z_k^2 - \frac{\sigma_{wk}}{2}\left\|\tilde{W}_k\right\|^2 - \frac{\sigma_{vk}}{2}\left\|\tilde{V}_k\right\|_F^2 - \frac{\sigma_{kk}}{2}\left\|\tilde{K}_k\right\|^2 + \xi_k\right\},$$

where ξ_k is as (15) except that

$$\xi_m = 0.2785\eta_m(\left\|V_m^*\right\|_F + \left\|W_m^*\right\| + \left\|W_m^*\right\|_1 + \varepsilon_{mU}) + \frac{\sigma_{wm}}{2}\left\|W_m^*\right\|^2 + \frac{\sigma_{vm}}{2}\left\|V_m^*\right\|_F^2 + \frac{\sigma_{km}}{2}\left\|K_m^*\right\|^2.$$

Choose $c_k > 0,\ \forall k = 1,\dots,m$, let

$$\varsigma = \min_{1\le k\le m}\left\{\frac{c_k}{0.5 g_{kU}}\right\} > 0,\ \delta = \sum_{k=1}^{m}\xi_k \ge 0$$

and choose

$$\sigma_{wk} \ge \varsigma\lambda_{\max}\left\{\Gamma_{wk}^{-1}\right\},\ \sigma_{vk} \ge \varsigma\lambda_{\max}\left\{\Gamma_{vk}^{-1}\right\},$$
$$\sigma_{kk} \ge \varsigma\lambda_{\max}\left\{\Gamma_{kk}^{-1}\right\}, \qquad k = 1,\dots,m.$$

We have

$$\dot{V}_m \le \sum_{k=1}^{m}\left\{-\varsigma 0.5 g_k^{-1} z_k^2 - \varsigma 0.5\tilde{W}_k^T \Gamma_{wk}^{-1}\tilde{W}_k - \varsigma 0.5 tr\left\{\tilde{V}_k^T \Gamma_{vk}^{-1}\tilde{V}_k\right\} - \varsigma 0.5\tilde{K}_k^T \Gamma_{kk}^{-1}\tilde{K}_k + \xi_k\right\} \le -\varsigma V_m + \delta.$$

To prove that the error trajectory is globally uniformly ultimately bounded, we proceed as follows. From Lemma 4.3 in [11], since we have $V_m > 0$ and radially unbounded, there exist class K_∞ functions, α_1, α_2 such that $\alpha_1(\|\bullet\|) \le V_m \le \alpha_2(\|\bullet\|)$. Choose some θ such that $0 < \theta < 1$. Then

$$\dot{V}_m \le -\theta\varsigma\alpha_1 - (1-\theta)\varsigma V_m + \delta.$$

Choosing $W = \theta\varsigma\alpha_1$ we see that

$$\dot{V}_m \le -W - (1-\theta)\varsigma V_m + \delta.$$

Therefore if $\left\|\left\{z_i, \tilde{W}_{fi}, \tilde{V}_{fi}, \tilde{W}_{gi}, \tilde{V}_{gi}, \tilde{K}_i\right\}\right\| \ge \mu$ where $\mu = \alpha_1^{-1}(\delta/(1-\theta)\varsigma)$ then

$$(1-\theta)\varsigma V_m \ge (1-\theta)\varsigma\alpha_1(\mu) = \delta.$$

Thus $\dot{V}_m \le -W$ for all $\left\|\left\{z_i, \tilde{W}_{fi}, \tilde{V}_{fi}, \tilde{W}_{gi}, \tilde{V}_{gi}, \tilde{K}_i\right\}\right\| \ge \mu.$ Therefore from Theorem 4.18 in [11], the trajectories $z_i, \tilde{W}_{fi}, \tilde{V}_{fi}, \tilde{W}_{gi}, \tilde{V}_{gi}, \tilde{K}_i$ are globally uniformly ultimately bounded. It can be shown that the ultimate bound is given by

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{2\Lambda_{\max}\delta}{\Lambda_{\min}^2(1-\theta)\varsigma}}, \tag{16}$$

where

$$\Lambda_{\max} = \max_{1\le k\le m}\left\{1, \lambda_{\max}(\Gamma_{wfk}^{-1}), \lambda_{\max}(\Gamma_{vfk}^{-1}), \lambda_{\max}(\Gamma_{wgk}^{-1}), \lambda_{\max}(\Gamma_{vgk}^{-1}), \lambda_{\max}(\Gamma_{kk}^{-1})\right\},$$

$$\Lambda_{\min} = \min_{1\le k\le m}\left\{1, \lambda_{\min}(\Gamma_{wfk}^{-1}), \lambda_{\min}(\Gamma_{vfk}^{-1}), \lambda_{\min}(\Gamma_{wgk}^{-1}), \lambda_{\min}(\Gamma_{vgk}^{-1}), \lambda_{\min}(\Gamma_{kk}^{-1})\right\}.$$

The exponential-decay upper bound of the error trajectories can be computed from $\dot{V}_m \le -\varsigma V_m + \delta$ to be

$$|z_1| \le \sqrt{2\left(\frac{\delta}{\varsigma} + (V_m(0))e^{-\varsigma t}\right)},\ \forall t \ge 0.$$

V. Simulation Results

Consider a system in strict feedback form:

$$\dot{x}_1 = 0.5x_1 + (1 + 0.1x_1^2)(x_2 - (2 + \sin x_1)),$$
$$\dot{x}_2 = x_1 x_2 + (2 + \cos x_1)(u - 0.3(e^{x_1} + e^{-x_1})),$$
$$y = x_1.$$

This system is similar to those used in [6]. Only output y is measurable and all nonlinear functions are unknown. The control objective is to make output y follow a smooth square wave. The design parameters are as follows:

(i) Sampling time is 0.01 sec. All initial values are set to zero.

(ii) Identifier parameters: $\lambda = 0.995$, number of fuzzy rules = 2, $P_0 = 10^6 I$,

for input x_1:
$c_1 = -0.67, c_2 = 0.67, a_1 = 0.67, a_2 = 0.67, b_1 = 1, b_2 = 1,$

for input x_2:
$c_1 = 1.33, c_2 = 2.67, a_1 = 0.67, a_2 = 0.67, b_1 = 1, b_2 = 1.$

(iii) Observer parameters: $\eta = 0.1,\ L = [10\ \ 20]^T$.

(iv) Controller parameters: number of nodes = 10, $\Gamma_{wi} = \Gamma_{vi} = 10, \Gamma_{ki} = 1, c_i = 15, \eta_i = 0.001,$

$\sigma_{wi} = \sigma_{vi} = \sigma_{ki} = 0.2,\ \forall i = 1,2.$

Overall control system performance is given in Fig. 4. Output is able to track the desired trajectory well. All states and control input are bounded.

VI. Conclusion

We have presented a model-free output feedback control system design for a nonlinear system in strict feedback form. Neuro-fuzzy systems are used as identifiers of the unknown plant functions. Observer objective is to estimate the actual states using actual plant output, plant input and estimated plant functions from the identifier. Controller is based on using three-layer neural networks to learn part of the desired backstepping control inputs. Simulation example shows good overall tracking performance.

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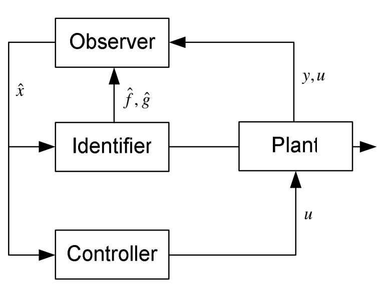

Fig. 1. Overall control system diagram.

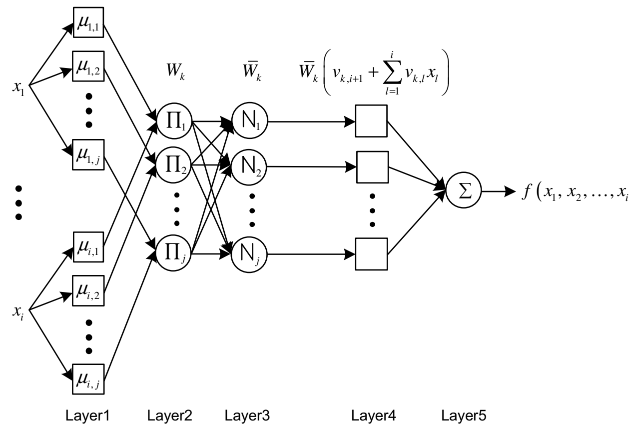

Fig. 2. Adaptive Neuro-Fuzzy Inference System (ANFIS).

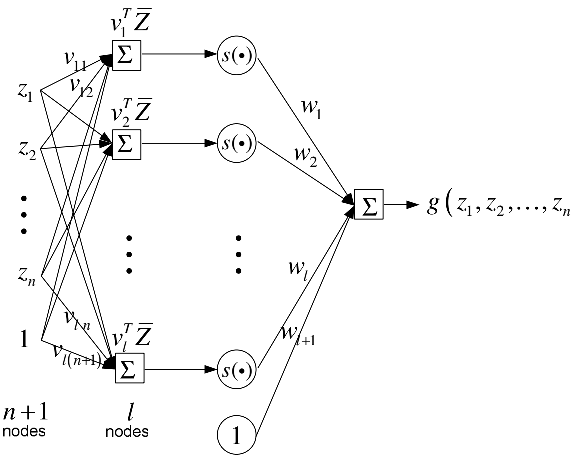

Fig. 3. A three-layer neural network.

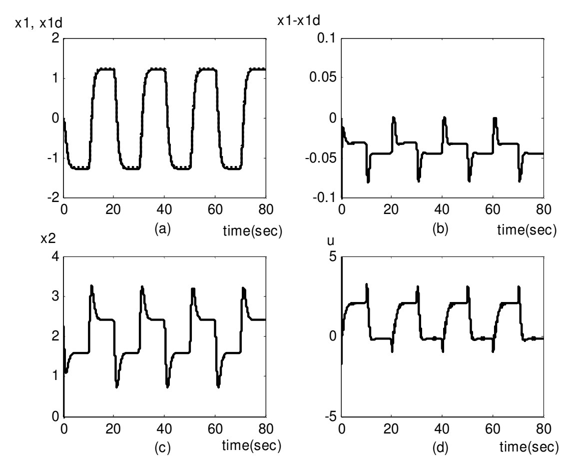

Fig. 4. Overall control system performance during 80 s. (a) Actual output, $y = x_1$, and desired output, $y_d = x_{1d}$. Dotted line is desired value and solid line is actual value. (b) Tracking error, $x_1 - x_{1d}$. (c) Actual state, x_2. (d) Actual control input u.