# Robust and Quadratic Stabilization of TORA System via Dynamic Surface Control Technique

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***Abstract***

This paper describes our research on robustly stabilizing the motion of *Translational Oscillator with Rotating Actuator (TORA) system* where mismatched uncertainties are present. The TORA system has fewer controls than the number of degrees of freedom, so-called *underactuated mechanical systems*, originally studied as a simplified model of a dual-spin spacecraft to investigate the resonance capture phenomenon. A novel recursive controller design associated with Control Lyapunov function method, namely, *Dynamic Surface Control (DSC)* is presented in this paper. The DSC is a robust control technique is generally applied to mismatched dynamic systems in strict feedback form to avoid the problem of an "explosion of terms" occurring in the integrator backstepping method. We use the method of global change of coordinates and collocated partial feedback linearization to transform the dynamic model of the TORA system into a strict feedback form, allowing us to successfully apply DSC. The uncertainties appear in the TORA system are handled by using the idea of nonlinear damping through the DSC. The performance of the proposed controller is compared with that of a controller designed using DSC with no nonlinear damping, referred as traditional DSC. Both the robust DSC and the traditional DSC perform well in the absence of disturbances. The proposed DSC is more robust when disturbances are introduced. Furthermore, for the given set of controller gains, the quadratic stability and also the region of attraction can be solved numerically.

***Keywords**:* Translational Oscillator with Rotating Actuator (TORA) system, Dynamic Surface Control, Underactuated Mechanical Systems, Robust Nonlinear Control.

## 1. Introduction

The problem of analysis and control of nonlinear mechanical systems has been an important research area for a long time. Due to the rich dynamical behavior of nonlinear systems causes to great advances in various nonlinear feedback control techniques such as Feedback Linearization [4], Passivity-Based Control (PBC) [7], [9], Sliding Mode Control (SMC) [3], Integrator Backstepping (IB) Control [5], [9], etc. However, by nature of Feedback Linearization which based on differential geometric theory in the 1980s; therefore, it cannot be used for tackle the system with uncertainty. There are much attention has been given recently about Lyapunov-based techniques e.g. SMC, Lyapunov redesign [5], IB, etc. As in the concept of energy dissipation is used for designing a control law via PBC [7]. For uncertain nonlinear systems, the SMC is one of the most effective tools, it has good tracking performance, but requires systems to meet matching condition of uncertainty. In other ways, the IB design procedure has the problem of "explosion of complexity" caused by the repeated differentiations of virtual controls.

An alternative control design method called Multiple Sliding Surface (MSS) control [14] was developed independently of IB but is mathematically very similar; however, MSS has the same problem as IB. In order to avoid the drawback of both IB and MSS above; a recursive and systematic controller design associated with Control Lyapunov function method, namely, Dynamics Surface Control (DSC) has been developed by Swaroop *et al*. [12]. The DSC, a novel control technique, can reduce the complexity of IB, and mathematical difficulty for the SMC analysis due to discontinuous functions, and to overcome its inability to mismatched uncertainty. The concept of DSC is introducing a first-order filtering of the synthesized virtual control law at each step of the MSS design procedure.

Underactuated Mechanical Systems (UMSs) are mechanical systems that have fewer control inputs than the number of degrees of freedom. Control of UMSs appears in a broad range of applications [6], including Robotics, Flexible Systems, Mobile Systems, Locomotive Systems, Marine Systems, and Aerospace Systems. In particular, Translational Oscillator with Rotating Actuator system, commonly referred to as TORA system, has been mainly a benchmark example of the UMSs, introduced by Wan *et al*. [13], and originally studied as a simplified model of a dual-spin spacecraft to investigate the resonance capture phenomenon. Global asymptotic stabilization of TORA system using output feedback has been known due to [1]. Various constructive nonlinear control methodologies have been tested on this system such as IB [5], [9], while in [7], the problem was set in an Euler-Lagrange framework and solved by the method of passivity-based output feedback control.

In this paper, we present the robust dynamic surface controller design procedure for the stabilization of the TORA system with mismatched uncertainties. The model uncertainties of the TORA are handled by using the concept of nonlinear damping assigned to the DSC. It is proved that the proposed design method is able to guarantee semi-global uniform ultimate boundedness of all signals in the closed-loop system, by appropriately choosing of controller gains and filter time constants. Finally, based on quadratic stability theory, feasibility of the fixed controller gains for quadratic stabilization can be tested and the region of attraction can be obtained.

The paper is organized as follows: In Section 2, we present the modeling of the TORA system by Euler-Lagrange method, then the method of collocated partial feedback linearization is used for transforming the TORA's state space model into strict feedback form. In Section 3, the DSC design and Lyapunov-based controller design are proposed for control the TORA system with mismatched uncertain. The stability, robustness, and performance of the proposed control system will be analyzed in Section 4. Simulation results are discussed and compared in Section 5. In Section 6 gives some conclusions.

## 2. Modeling of the TORA System

The TORA system as illustrated in Fig. 1, where a translational platform of mass $m_1$ is stabilized by an eccentric rotational mass $m_2$ located at a distance $r$ from the platform's center of mass. The oscillator platform is connected to a fixed support via a linear spring of stiffness $k_1$, constrained to one-dimensional motion. By neglecting of model uncertainties, the dynamic model of the TORA system can be obtained by using Euler-Lagrange method.

### 2.1 Euler-Lagrange Equation for TORA system

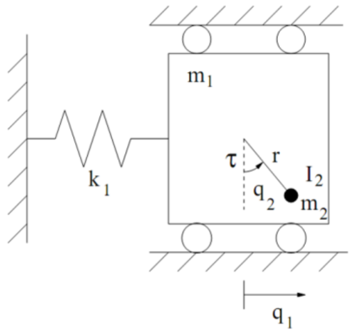



Fig. 1 TORA system configuration [6].

The Lagrangian for TORA system is written as

$$\begin{aligned} L(q,\dot{q}) \;=\; & \tfrac{1}{2} m_1 \dot{q}_1^2 + \tfrac{1}{2} m_2 \Big[ (\dot{q}_1 + r\cos(q_2)\dot{q}_2)^2 \\ & + (r\sin(q_2)\dot{q}_2)^2 \Big] + \tfrac{1}{2}(m_2 r^2 + I_2)\dot{q}_2^2 \\ & - \tfrac{1}{2} k_1 q_1^2 - m_2 g r\cos(q_2) \end{aligned} \qquad (1)$$

and $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q(q)u$ gives equations of motion

$$\begin{bmatrix} m_1 + m_2 & m_2 r\cos(q_2) \\ m_2 r^2 \cos(q_2) & m_2 r^2 + I \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -m_2 r\sin(q_2)\dot{q}_2^2 + k_1 q \\ m_2 g r\sin(q_2) \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \qquad (2)$$

### 2.2 Collocated Partial Feedback Linearization

The state space model of the TORA system is unsuitable for direct application of DSC method, since they are not being in strict feedback form [5]. The procedure of collocated partial feedback linearization was developed by Spong [11]. It employs a change of control and coordinate transformation to change the original equation of motion of the TORA system into a system in cascade form. After collocated partial feedback linearization using a change of control via the state feedback

$$\tau = \alpha(q_2)u + \beta(q,\dot{q}) \qquad (3)$$

with

$$\alpha(q_2) = (m_2 r_2^2 + I_2) - \frac{(m_2 r\cos(q_2))^2}{(m_1 + m_2)} > 0,\ \ \forall q_2 \in [-\pi,\pi]$$

and

$$\beta(q,\dot{q}) = m_2 g r\sin(q_2) + m_2 r\cos(q_2)\frac{(m_2 r\sin(q_2)\dot{q}_2^2 + k_1 q_1)}{(m_1 + m_2)}$$

where $u$ is the new control input. Thus, based on Theorem 4.2.1 [6], the global change of coordinates

$$\begin{aligned} q_r &= q_1 + (m_2 r \sin(q_2))/(m_1+m_2) \\ p_r &= (m_1+m_2)p_1 + m_2 r \cos(q_2) p_2 \end{aligned} \quad (4)$$

transforms the dynamics of the TORA system into a cascade nonlinear system in strict feedback form

$$\begin{aligned} \Sigma_{Core} &: \begin{cases} \dot{q}_r = \overline{m} p_r \\ \dot{p}_r = -k_1 q_r + \bar{\varepsilon} \sin(q_2) \end{cases} \\ \Sigma_{Outer} &: \begin{cases} \dot{q}_2 = p_2 \\ \dot{p}_2 = u \end{cases} \end{aligned} \quad (5)$$

where

$$\overline{m} := (m_1+m_2)^{-1} \quad \text{and} \quad \bar{\varepsilon} := \frac{k_1 m_2 r}{(m_1+m_2)}$$

**2.3 TORA System with Mismatched Uncertainties**

We augment the plant uncertainties $\Delta f_1$ and $\Delta f_2$ in the nominal TORA system (5) as mismatched uncertainties in the sense that they cannot be controlled directly by the control input $u$. Consequently, the uncertain nonlinear TORA system is given by

$$\begin{aligned} \Sigma_{Core} &: \begin{cases} \dot{q}_r = \overline{m} p_r + \Delta f_1 \\ \dot{p}_r = -k_1 q_r + \bar{\varepsilon} \sin(q_2) + \Delta f_2 \end{cases} \\ \Sigma_{Outer} &: \begin{cases} \dot{q}_2 = p_2 \\ \dot{p}_2 = u \end{cases} \end{aligned} \quad (6)$$

These uncertainties are unknown but bounded by $|\Delta f_1(q_r)| \le \rho_1(q_r)$ and $|\Delta f_2(q_r, p_r)| \le \rho_2(q_r, p_r)$ with $\rho_1(0)=0$ and $\rho_2(0,0)=0$. Moreover, each of components of $\Delta f$ is locally Lipschitz nonlinearity to guarantee the existence and uniqueness of the solution of (6).

However, the Core-subsystem $(\Sigma_{Core})$ in (6) has an implicit term $\sin(q_2)$ that cannot directly design a virtual input $q_2$ to handle the uncertainty $\Delta f_2$ in the DSC procedure. To cope with the problem, we use the change of variables

$$x_1 = q_r,\ x_2 = p_r,\ x_3 = \sin(q_2),\ \text{and} \ \ x_4 = u$$

yield

$$\begin{aligned} \Sigma_{Core} &: \begin{cases} \dot{x}_1 = \overline{m} x_2 + \Delta f_1 \\ \dot{x}_2 = -k_1 x_1 + \bar{\varepsilon} x_3 + \Delta f_2 \end{cases} \\ \Sigma_{Outer} &: \begin{cases} \dot{x}_3 = x_4 \cos(\arcsin(x_3)) \\ \dot{x}_4 = u \end{cases} \end{aligned} \quad (7)$$

Now the state space equation (7) is in strict feedback form; moreover, its Core-subsystem $(\Sigma_{Core})$ has explicitly in a virtual input $x_3$ allows us to design a robust DSC for tackling those uncertainties. However, the implicit term $\sin(q_2)$ cannot be completely deleted from the system. A Lyapunov-based controller design will be used later for the Outer-subsystem $(\Sigma_{Outer})$.

**3. Robust Controller Design**

The overall closed-loop control system is summarized and shown in the Fig. 2. It consists three parts as follows: First, a collocated partially feedback linearization block. Second, a TORA (or nominal) system with its uncertainty block. Third, the controller block which it contains two different control methods i.e., DSC and Lyapunov-based control.

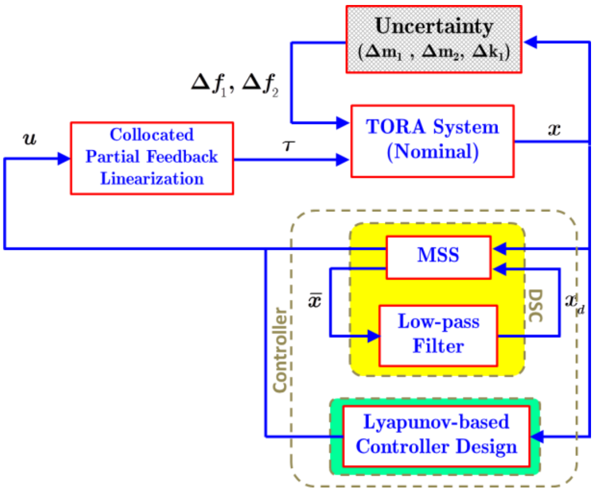



Fig. 2 Overall Control System Architecture

The control objective is to stabilize the cart's position and its velocity $(q_1 = 0, \dot{q}_1 = 0)$, which corresponds to make $q_r = 0$, $p_r = 0$ when the system started from some of an initial condition.

**3.1 DSC Design for Core-Subsystem**

Let the 1st-sliding surface $S_1 := x_1$. After differentiating $S_1$ and using (7),

$$\dot{S}_1 = \overline{m} x_2 + \Delta f_1$$

Then, the 2nd-sliding surface is defined as $S_2 := x_2 - x_{2d}$ where $x_{2d}$ called the virtual (or synthetic) input received from the output of the 1st-order low-pass filter:

$$\tau_2 \dot{x}_{2d} + x_{2d} = \bar{x}_2, \quad x_{2d}(0) = \bar{x}_2(0) \tag{8}$$

where $\tau_2$ is the filter time constant and $\bar{x}_2$ will be design to drive $S_1 \to 0$. The derivative of $S_2$ is

$$\dot{S}_2 = \dot{x}_2 - \dot{x}_{2d} = -k_1 S_1 + \bar{\varepsilon} x_3 + \Delta f_2 - \dot{x}_{2d}$$

Thus, the core-subsystem $(\Sigma_{Core})$ in (7) can be written in the term of $S_1$, $S_2$, $x_{2d}$, and $x_{3d}$ as follows

$$\Sigma_{Core}: \begin{cases} \dot{S}_1 = \bar{m}(S_2 + x_{2d}) + \Delta f_1 \\ \dot{S}_2 = -k_1 S_1 + \bar{\varepsilon}(S_3 + x_{3d}) + \Delta f_2 - \dot{x}_{2d} \end{cases} \tag{9}$$

where $S_3 := x_3 - x_{3d}$ is the 3rd-sliding surface, considered as an input for (9) while $x_{3d}$ will be design to drive $S_2 \to 0$.

By using Lyapunov's method and Young's inequality:

$$S_i \Delta f_i \le |S_i| |\Delta f_i| \le \rho_i |S_i| \le \tfrac{\varepsilon_i}{2} + \tfrac{S_i^2 \rho_i^2}{2\varepsilon_i} \tag{10}$$

where $\varepsilon_i$ is arbitrary. The stabilizing function $\bar{x}_2$ and $x_{3d}$ can be designed as follows

For $\bar{x}_2$ with respect to the Lyapunov function candidate $V_1(S_1) = S_1^2/2$, we assume that $\bar{x}_2 \to x_{2d}$ and $x_2 \to x_{2d}$ (i.e., $S_2 \to 0$) and using the idea of nonlinear damping [5]. The following choice of $\bar{x}_2$ is as follows

$$\bar{x}_2 := -\tfrac{1}{\bar{m}}\left(K_1 S_1 + \tfrac{\rho_1^2}{2\varepsilon_1} S_1\right) \tag{11}$$

with $K_1 > 0$, resulting in

$$\begin{aligned} \dot{V}_1 = S_1 \dot{S}_1 &= -\left(K_1 + \tfrac{\rho_1^2}{2\varepsilon_1}\right) S_1^2 + S_1 \Delta f_1 \\ &\le -K_1 S_1^2 + \varepsilon_1/2 \end{aligned}$$

Therefore, $S_1$ is uniformly ultimately bounded.

Similarly, for $x_{3d}$, setting $V_2(S_2) = S_2^2/2$ as a Lyapunov function candidate and assuming that $x_3 \to x_{3d}$ (i.e., $S_3 \to 0$) and also using the idea of nonlinear damping [5]. A reasonable choice for $x_{3d}$ would be to set

$$x_{3d} := \tfrac{1}{\bar{\varepsilon}}\left(\dot{x}_{2d} + k_1 S_1 - K_2 S_2 - S_2 \tfrac{\rho_2^2}{2\varepsilon_2}\right) \tag{12}$$

with $K_2 > 0$, resulting in

$$\begin{aligned} \dot{V}_2 = S_2 \dot{S}_2 &= S_2\left(-K_2 S_2 - \tfrac{\rho_2^2}{2\varepsilon_2} + \Delta f_2\right) \\ &\le -(K_2 - \bar{\varepsilon}/2) S_2^2 + \varepsilon_2/2 \end{aligned}$$

Therefore, $S_2$ is uniformly ultimately bounded.

*Remark:* We can see that the states $(S_1, S_2)$ are forced to the origin via a virtual (or synthetic) input $x_3$ from the Outer-subsystem $(\Sigma_{Outer})$.

### 3.2 Lyapunov-based Design for Outer-Subsystem

The aim of using the change of variable in Section 2.3 is in order to can apply the designing of DSC for the uncertain Core-subsystem $(\Sigma_{Core})$. However, the DSC methodology cannot be used for the Outer-subsystem $(\Sigma_{Outer})$, because there is a cosine function multiplying with $x_4$ such that the error filter's dynamics cannot formed in a linear equation. Therefore, a basic nonlinear controller design associated with a suitable Lyapunov function is used for the Outer-subsystem.

To drive $x_3 \to x_{3d}$, we define the 3rd-sliding surface $S_3 := x_3 - x_{3d}$ and let $S_4 := x_4$. The Outer-subsystem $(\Sigma_{Outer})$ in (7) can be written in the term of $S_3$, $S_4$, $x_{3d}$, and $x_{4d}$ as follows

$$\Sigma_{Outer}: \begin{cases} \dot{S}_3 = (S_4 + x_{4d})\cos(\arcsin(S_3 + x_{3d})) - \dot{x}_{3d} \\ \dot{S}_4 = u \end{cases} \tag{13}$$

Consider $V_3(S_3, S_4) = \left(S_3^2 + S_4^2\right)/2$ as a Lyapunov function candidate. A control $u$ is found such that $\dot{V}_3$ is rendered negative definite. Trying

$$u := -K_3 S_3 - c_1 \arctan(c_2 S_4) \tag{14}$$

with $K_3 > 0$, resulting in

$$\dot{V}_3 = -\left(\tfrac{K_3 - 1}{2}\right)\left(S_3^2 + S_4^2\right) - c_1 S_4 \arctan(c_2 S_4) < 0$$

where $K_1, K_2$, and $K_3$ are controller gains, and are determined later.

## 4. Stability Analysis

In section 3, we separately designed the stabilizing functions (or virtual controls) for the Core-subsystem, and designed the control input $u$ for the Outer-subsystem by using any type of Lyapunov functions candidate. However, the dynamics of 1st-order LPF (8) occurring in DSC procedure, is included in the overall system for eliminates "the explosion of term", does not proved stability property.

### 4.1 Stability of Augmented Error Dynamics

After defining the filter error $\xi_2 := x_{2d} - \bar{x}_2$, and including its dynamics from (8), the augmented closed-loop error dynamics are

$$\begin{cases} \dot{S}_1 = -K_1 S_1 + \overline{m}(S_2 + \xi_2) - S_1 \frac{\rho_1^2}{2\varepsilon_1} + \Delta f_1 \\ \dot{S}_2 = -K_2 S_2 - \bar{\varepsilon} S_3 - S_2 \frac{\rho_2^2}{2\varepsilon_2} + \Delta f_2 \\ \dot{\xi}_2 = \dot{x}_{2d} + \frac{d}{dt}\left(\frac{K_1}{\overline{m}} S_1 + \frac{\rho_1^2}{2\varepsilon_1} S_1\right) := -\frac{\xi_2}{\tau_2} + \eta_1(S_1, S_2, \xi_2) \\ \dot{S}_3 = S_4 \cos(\arcsin(S_3 + x_{3d})) - \dot{x}_{3d} \\ \quad := S_4 \cos(\arcsin(S_3 + x_{3d})) + \eta_2(S_1, S_2, \xi_2) \\ \dot{S}_4 = -K_3 S_3 - c_1 \arctan(c_2 S_4) \end{cases} \tag{15}$$

where $\eta_1$ and $\eta_2$ is a nonlinear function of $S_1, S_2$ and $\xi_2$.

To determine stability, suppose the Lyapunov function candidate be

$$V(x) = \left(S_1^2 + S_2^2 + \xi_2^2 + S_3^2 + S_4^2\right)/2$$

and its derivative of $V$ along the trajectory of (15) is

$$\begin{aligned} \dot{V} &= S_1\dot{S}_1 + S_2\dot{S}_2 + \xi_2\dot{\xi}_2 + S_3\dot{S}_3 + S_4\dot{S}_4 \\ &= -K_1 S_1^2 + \overline{m} S_1 S_2 + \overline{m}\xi_2 S_1 - S_1^2 \frac{\rho_1^2}{2\varepsilon_1} + S_1 \Delta f_1 \\ &\quad - K_2 S_2^2 - \bar{\varepsilon} S_2 S_3 - S_2^2 \frac{\rho_2^2}{2\varepsilon_2} + S_2 \Delta f_2 \\ &\quad - \frac{1}{\tau_2}\xi_2^2 + \xi_2 \eta_1 + S_3 S_4 \cos(\arcsin(S_3 + x_{3d})) \\ &\quad + S_3 \eta_2 - K_3 S_3 S_4 - c_1 S_4 \arctan(c_2 S_4) \end{aligned}$$

$$\begin{aligned} \dot{V} \le\ & \frac{\overline{m}}{2}\left(2S_1^2 + S_2^2 + \xi_2^2\right) + \frac{\bar{\varepsilon}}{2}\left(S_2^2 + S_3^2\right) - K_1 S_1^2 \\ & - K_2 S_2^2 + \left(\varepsilon + \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2}\right) - \frac{\xi_2^2}{\tau_2} + \frac{\xi_2^2 \eta_1^2}{2\varepsilon} \\ & + \frac{S_3^2 \eta_2^2}{2\varepsilon} + \left(\frac{1-K_3}{2}\right)\left(S_3^2 + S_4^2\right) - \underbrace{c_1 S_4 \arctan(c_2 S_4)}_{>0\ \forall S_4} \end{aligned} \tag{16}$$

where the last inequality comes from Young's inequality (10) for $i = 1, 2$ and

$$\xi_2 \eta_1 \le \frac{\varepsilon_0}{2} + \frac{\xi_2^2 \eta_1^2}{2\varepsilon_0}, \quad S_3 \eta_2 \le \frac{\varepsilon_0}{2} + \frac{S_3^2 \eta_2^2}{2\varepsilon_0}$$
$$S_1 \xi_2 \le \frac{1}{2}\left(S_1^2 + \xi_2^2\right), \quad S_i S_{i+1} \le \frac{1}{2}\left(S_i^2 + S_{i+1}^2\right)$$
$$\text{for } i = 1, 2, 3.$$

Consider the compact (closed and bounded) and convex set

$$D = \{ z \in \Re^5 \mid S_1^2 + S_2^2 + \xi_2^2 + S_3^2 + S_4^2 \le 2p \},$$

where $p > 0$, $z = [S_1\ S_2\ \xi_2\ S_3\ S_4]^T$. Then, there exits positive constants $M_1 > 0$ and $M_2 > 0$ such that $\eta_1$ and $\eta_2$ are bounded; i.e., $|\eta_1| \le M_1$ and $|\eta_2| \le M_2$ on $D$. For the surface gains $K_1 = K_2 := \overline{m}(2 + K_0)$, $K_3 = 2 + \bar{\varepsilon} + M_2^2/\varepsilon_0$ where $K_0 > \bar{\varepsilon}/2\overline{m}$, and choose the time constant such that

$$\frac{1}{\tau_2} = 2\overline{m} + \frac{M_1^2}{2\varepsilon_0} + K_0 \overline{m}$$

Then the inequality (16) is written as

$$\begin{aligned} \dot{V} \le\ & \frac{\overline{m}}{2}\left(2S_1^2 + S_2^2 + \xi_2^2\right) + \frac{\bar{\varepsilon}}{2}\left(S_2^2 + S_3^2\right) \\ & - \left(\overline{m}(2 + K_0)\right)\left(S_1^2 + S_2^2\right) + \left(\varepsilon_0 + \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2}\right) \\ & - \xi_2^2\left(2m + \frac{M_1^2}{2\varepsilon_0} + K_0 \overline{m}\right) + \frac{\xi_2^2 M_1^2}{2\varepsilon_0} \frac{\eta_1^2}{M_1^2} \\ & + \frac{S_3^2 \eta_2^2}{2\varepsilon_0} - \left(\frac{1 + \bar{\varepsilon} + M_2^2/\varepsilon_0}{2}\right)\left(S_3^2 + S_4^2\right) \\ \le\ & -2\overline{m}K_0\left(S_1^2 + S_2^2 + \xi_2^2\right) - \frac{1}{2}\left(S_3^2 + S_4^2\right) \\ & + \underbrace{\left(\varepsilon_0 + \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2}\right)}_{:=\varepsilon} - \left(1 - \frac{\eta_1^2}{M_1^2}\right)\frac{M_1^2 \xi_2^2}{2\varepsilon_0} - \left(1 - \frac{\eta_2^2}{M_2^2}\right)\frac{M_2^2 S_3^2}{2\varepsilon_0} \end{aligned}$$

Therefore,

$$\dot{V} \le -2\overline{m}K_0\left(S_1^2 + S_2^2 + \xi_2^2\right) - \frac{1}{2}\left(S_3^2 + S_4^2\right) + \varepsilon$$

We conclude that the solutions are *globally uniformly ultimately bounded*. Since $\varepsilon$ is arbitrary, the *ultimately error bound* can be made arbitrarily small.

### 4.2 Norm-Bounded Error Dynamics and Quadratic Stabilization for $q_r$-$p_r$

The stability proof and determination of the controller gains for the augmented closed-loop error dynamics as shown in Section 4.1 is not straightforward, since it requires the upper bound of the nonlinear functions $\eta_1$ and $\eta_2$ . In this section, their stability and region of attraction will be considered.

Since, only the Core-subsystem is designed by DSC. By assuming $S_3 \to 0$, the closed-loop error dynamics of the Core-subsystem (13) is written as follows

$$\begin{cases} \dot{S}_1 := -K_1 S_1 + \overline{m}(S_2 + \xi_2) + d_1 := C_z z + d_1 \\ \dot{S}_2 := -K_2 S_2 + d_2 \\ -\frac{K_1}{\overline{m}}\dot{S}_1 - \dot{\xi}_2 = -\frac{\xi_2}{\tau_2} + \frac{\partial}{\partial S_1}\left(\frac{K_1}{\overline{m}} S_1 + \frac{\rho_1^2}{2\varepsilon_1}\right)\dot{S}_1 \\ \qquad := -\frac{\xi_2}{\tau_2} + \phi c_z z + \phi d_1 \end{cases} \quad (17)$$

where the error state $z = [S_1 \ S_2 \ \xi_2]^T \in \Re^3$,

$C_z := [-K_1 \ \overline{m} \ \overline{m}] \in \Re^{1\times 3}$, $d_1 := \Delta f_1 - S_1 \frac{\rho_1^2}{2\varepsilon_1}$,

$d_2 := \Delta f_2 - S_2 \frac{\rho_2^2}{2\varepsilon_2}$, and $\phi := \frac{K_1}{\overline{m}} + \frac{\rho_1^2}{2\varepsilon_1} + \frac{S_1\rho_1}{\varepsilon_1}\frac{\partial \rho_1}{\partial S_1}$

Equation (17) can be rewritten in matrix form as

$$T\dot{z} = A_z z + \overline{B}_w w + \overline{B}_{n1} n_1 + \overline{B}_{n2} n_2 \quad (18)$$

where $w = \phi C_z z \in \Re^1$, $n_1 = [d_1 \ \phi_1 d_1]^T \in \Re^2$,

$$n_2 = [d_2 \ \phi_2 d_2]^T \in \Re^2,\ T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-K_1}{\overline{m}} & 0 & 1 \end{bmatrix},$$

$$A_z = \begin{bmatrix} -K_1 & \overline{m} & \overline{m} \\ 0 & -K_2 & 0 \\ 0 & 0 & -\frac{1}{\tau_2} \end{bmatrix},\ \overline{B}_w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\overline{B}_{n1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and}\ \overline{B}_{n2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, $T$ is always invertible. After multiplying the inverse matrix $T^{-1}$ to both sides in (18), the Core-subsystem's augmented error dynamics are rewritten as

$$\begin{cases} \dot{z} = A_{cl} z + B_w w + B_{n1} n_1 + B_{n2} n_2 \\ w = \phi C_z z \end{cases} \quad (19)$$

where $A_{cl} = \begin{bmatrix} -K_1 & \overline{m} & \overline{m} \\ 0 & -K_2 & 0 \\ \frac{-K_1^2}{\overline{m}} & K_1 & K_1 - \frac{1}{\tau_2} \end{bmatrix}$,

$$B_w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ B_{n1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \frac{K_1}{\overline{m}} & 1 \end{bmatrix},\ \text{and}\ B_{n2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, an inequality constraint for $w$ can be imposed on a convex set $D_i \subset D$. Since $f_1$ and $\rho_1$ are in $C^1$ -function, there exists a Lipschitz constant $\gamma > 0$ such that $|\phi| \le \gamma$ . Then,

$$\| w \| \le \gamma \| C_z z \| := \| \tilde{C}_z z \|$$

For the given controller gains $K_1, K_2$ and $\tau_2$ . The augmented closed-loop error dynamics (19) is formed in a special class of linear differential inclusions (LDIs) called norm-bounded LDIs (NLDIs) [2], can be regarded as a linear system subject to a vanishing perturbation $w$ and the nonvanishing perturbations $n_1$ and $n_2$ . The following definition and theorem are given for the meaning of *quadratically stable* [10].

*Definition 1* Let $z = 0$ be an exponentially stable equilibrium point of the nominal system, $\dot{z} = A_{cl} z$ when $A_{cl}$ is Hurwitz for the given set of controller gains, $\Theta = \{K_1, K_2, \tau_2\}$ . Then, a nominal nonlinear system is *quadratic stabilizable* via DSC if there exist a positive definite matrix $P$ such that

$$\dot{V}(z) = \tfrac{d}{dt}(z^T P z) = (A_{cl} z + B_w w)^T P z + z^T P (A_{cl} z + B_w w) < 0 \qquad \square$$

We are interested in finding a quadratic Lyapunov function. If a quadratic Lyapunov function exists for this system, then the system is said to be *quadratically stable*.

*Theorem 1* Suppose that the closed-loop error dynamics (19) is given for the given set of controller gains, $\Theta = \{K_1, K_2, \tau_2\}$ . If $A_{cl}$ is Hurwitz, i.e., there exist $P > 0$ and $Q = Q^T > 0$ such that

$$PA_{cl} + A_{cl}^T P = -Q \quad (20)$$

and $\gamma < \lambda_{\min}(Q)/(2\lambda_{\max}(P)\| B_w C_z \|_2)$ for $D_i = \{x \in \Re^n \mid \| J \| \le \gamma\} \subset D$ where $J$ is Jacobian matrix $J = [\partial f / \partial x]$, the origin in (17) is *exponentially stable* on $D_i$ . Thus a nominal nonlinear system is quadratically stabilizable via DSC with the gain $\Theta$ on $D_i$. Furthermore, $D_i$ is the region of attraction. □

*Proof*: see [10].

Applying Theorem 1 for the convex set $D_i = \{\, x \in \Re^2 \mid |x_1| \le \gamma/2 \,\}$. Suppose the set of controller gains is

$$\Theta = \{K_1, K_2, \tau_2\} = \{0.1, 0.1, 0.1\}$$

Then $\lambda(A_{cl}) = \{-0.1010, -0.1010, -9.8990\}$ is Hurwitz and the solution of (20) for $Q = I_3$ is

$$P = \begin{bmatrix} 4.9511 & 2.2609 & 0.0445 \\ 2.2609 & 7.0768 & 0.0215 \\ 0.0445 & 0.0215 & 0.0509 \end{bmatrix} > 0$$

which has $\lambda_{\max}(P) = 8.5125$. Consequently, according to Theorem 1, the origin of (7) is exponentially stable if

$$\gamma < 1/\left(2\lambda_{\max}(P) \| B_w C_z \|_2\right) = 0.3606$$

That is; the $(S_1, S_2, \xi_2)$ dynamics is exponentially stable if $2|x_1| < 0.3606$. Hence, we can define a region of attraction as the domain $D_i = \{x \in \Re^2 \mid |x_1| \le 0.1803\}$. Finally, the simulation result for all error surfaces shown in Fig. 3

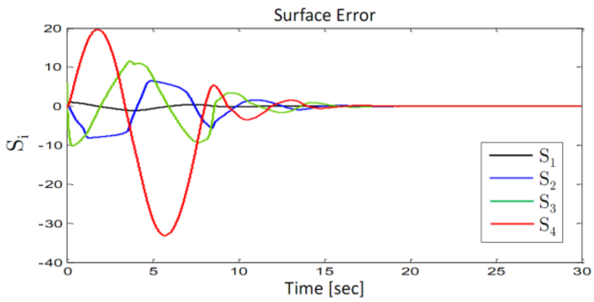



Fig. 3 Time responses of all error surfaces

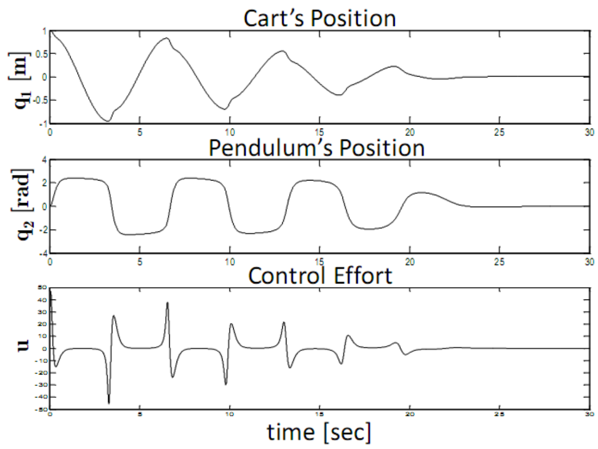
## 5. Simulation Results

The performance of the proposed robust controller is compared with a conventional DSC. The physical parameters also an initial condition of the TORA system have been kept same as by Olfati-Saber [6] and Qaiser [8], i.e., $m_1 = 10$, $m_2 = 1$, $k_1 = 10$, $r = 1$, $I_2 = 10$, and initial condition would be set as $x(0) = [1\,0\,0\,0]^T$. The parameters for the proposed robust DSC are determined as follows $K_1 = 0.1$, $K_2 = 0.1$, $K_3 = 2$, $\tau_2 = 0.1$, $c_1 = 2$, $c_2 = 1.3$, $\rho_1 = \rho_2 = 0.2$, and $\varepsilon_1 = \varepsilon_2 = 0.5$.

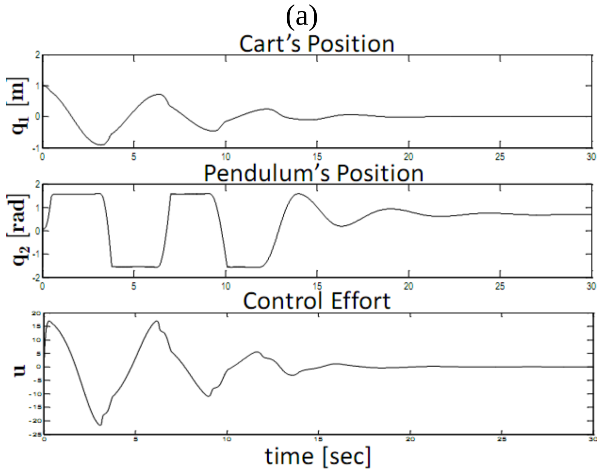
*Case I*: TORA system with no uncertainty model $(\Delta f_i = 0)$. The conventional DSC designed by Qaiser [8] was simulated (Fig. 4(a)) and compared with the proposed robust DSC (Fig. 4(b)). The performance of both controllers has satisfactory response for exact model. Moreover, the proposed DSC used the less control effort.

In order to verify the robust performance of proposed robust DSC. The uncertain term is given by $\Delta f_i = 0.17 S_i^2 \sin(0.1 S_i^2)$ for $i = 1, 2$ and used for testing of the both controllers.

*Case II*: TORA system with uncertainty model $(\Delta f_i \ne 0)$. The simulation result with conventional DSC is illustrated in Fig. 5(a). It can be seen that the conventional DSC cannot tackle with that uncertainty. For this case, the states $q_1, q_2$ diverge and the control $u$ is unbounded when time goes to infinity (In practically, about 100sec). However, the proposed robust DSC is utilized but more of the control effort to deal in this case.

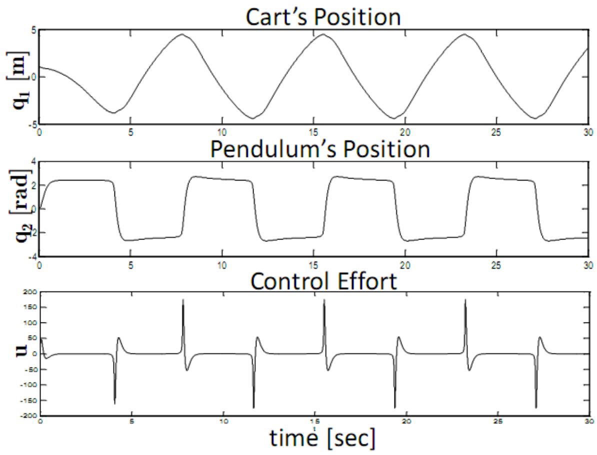



(a)

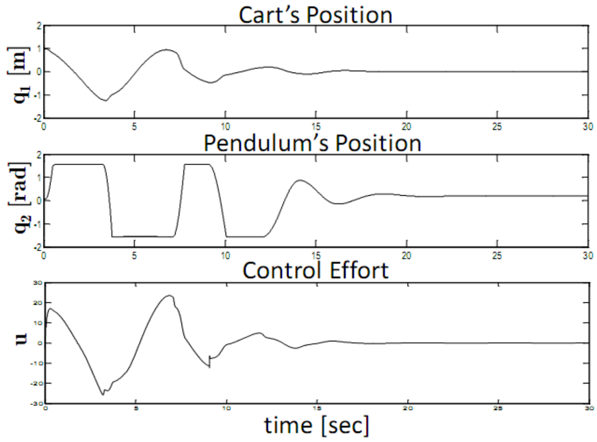



(b)

Fig. 4 Simulation results for the unperturbed TORA system: (a) traditional DSC and (b) robust DSC




(a)




(b)

Fig. 5 Simulation results for the perturbed TORA system: (a) traditional DSC and (b) robust DSC

## 6. Conclusions

The original TORA's state space model is transformed into strict feedback form which allows us for designing with DSC method. Moreover, the implicit term $\sin(x_3)$ as appearing in the Core-subsystem is eliminated by using the change of variables such that the proposed DSC can be applied on the perturbed $x_1 - x_2$ dynamics. The two controller techniques; first is traditional DSC and second is robust DSC with a nonlinear damping, were designed and simulated for the TORA system with mismatched uncertainty. Both are perform well in the absence of disturbances. The proposed DSC is more robust when a non-Lipschitz uncertainty which can be considered as disturbances are introduced, but it use more control effort in this case. Furthermore, the stability of the system is analyzed even in theoretically and numerically. Future improvements envisioned include enhancing the robust the robustness and generalization of the DSC scheme to the whole subclass of UMS.

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